

ASYMPTOTICS OF SOLUTIONS OF A PARABOLIC EQUATION NEAR SINGULAR POINTS

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ABSTRACT. Results of investigation of the asymptotic behavior of solutions to the Cauchy problems for a quasi-linear parabolic equation with a small parameter at a higher derivative near singular points of limit solutions are presented. Interest to the problem under consideration is explained by its applications to a wide class of physical systems and probabilistic processes such as acoustic waves in fluid and gas, hydrodynamical turbulence and nonlinear diffusion. The following cases are considered: a singularity generated by a jump discontinuity of the initial function, collision of two shock waves, gradient catastrophe, transition of a weak discontinuity into a shock wave, a singularity generated by a large initial gradient.

1 Introduction.

A simplest model of the motion of continuum, which takes into account nonlinear effects and dissipation, is the equation of nonlinear diffusion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}$$

for the first time presented by J. Burgers [1]. This equation is used in studying the evolution of a wide class of physical systems and probabilistic process, acoustic waves in fluid and gas [7].

In the present survey, results of investigations of the asymptotic behavior of solutions to the Cauchy problems for a more general quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad t \geq t_0, \quad (1)$$

$$u(x, t_0) = q(x), \quad x \in \mathbb{R}, \quad (2)$$

are collected. We assume that $\varepsilon > 0$, the function φ is infinitely differentiable and its second derivative is strictly positive. The initial function q is bounded and piecewise smooth.

Such models are constructed for studying the motion of motor transport, flood waves and others [2]. The interest to problem under consideration is explained by physical applications and the fact that its solutions allow one to obtain viscous generalized solutions of the limit equation. This problem had been studied by N.S. Bakhvalov, I.M. Gelfand, A.M. Il'in, E. Hopf, O.A. Ladyzhenskaya, O.A. Oleinik, and many other mathematicians. It is strictly proved [5], that there exists a unique bounded infinitely differentiable with respect to x and t solution $u(x, t, \varepsilon)$.

In this survey, we present results of investigation of the asymptotic behavior of solutions to the Cauchy problems in neighborhoods of singular points, which arise on discontinuity sets of the limit (degenerate) solution with $\varepsilon = 0$. In the next sections the following problems are considered.

1. *Jump discontinuity of the initial function.* A.M. Il'in and T.N. Nesterova [3] considered problem (1)–(2) in the case when the initial function has a finite jump discontinuity.

2. *Collision of two shock waves.* In the same paper [3], problem (1)–(2) is considered in the case when the limit solution on a finite interval of time has two smooth curves of discontinuity $x = s_1(t)$ and $x = s_2(t)$, merging at the moment $t = t^*$ into one $x = s_3(t)$. For the Burgers equation this case is given in Whitham's book [7].

3. *Gradient catastrophe.* In [4], A.M. Il'in studied the case when in the strip $\{(x, t) : t_0 \leq t \leq T, x \in \mathbb{R}\}$ the limit ($\varepsilon = 0$) solution of the problem is smooth everywhere except for one smooth discontinuity curve. This problem in detail is considered in his classical monograph [2].

4. *Transition of a weak discontinuity into a shock wave.* In [6], V.G. Sushko studied problem (1)–(2) in the case when the initial function $q(x)$ is smooth everywhere except for one point, at which it is continuous, while the first derivative has a jump discontinuity. Such a weak discontinuity in the limit problem propagates along a characteristic line for a finite time and then becomes a shock wave. The asymptotics of the solution in a small parameter ε in a neighborhood of the transition point is obtained in [8, 9].

5. *Singularity generated by a large initial gradient.* In papers [10, 11], the author studied the problem with two small parameters: a viscosity (diffusion) parameter ε at a higher derivative and an additional parameter ρ in the initial condition $u(x, t_0) = \nu(x/\rho)$.

Application of the matching method to investigation of solutions of the above problems leads to the necessity of constructing several asymptotic series in distinct subdomains of independent variables. For this reason, these asymptotics are called singular and such problems are classified as singularly perturbed.

2 Discontinuity of the initial function.

In papers by A.M. Il'in and T.N. Nesterova [3], the problem is considered in the case when the initial function has a finite jump discontinuity. Assuming, without loss of generality, that the jump of the function u_0 lies at the point $x = 0$ and the initial moment of time is $t_0 = 0$, it is convenient to introduce the inner variables

$$\zeta = \frac{x - s(t)}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon},$$

where $x = s(t)$ is a curve of discontinuity of the limit solution. Then equation (1) becomes

$$\frac{\partial^2 w}{\partial \zeta^2} + s'(\varepsilon\tau) \frac{\partial w}{\partial \zeta} - \frac{\partial \varphi(w)}{\partial \zeta} - \frac{\partial w}{\partial \tau} = 0.$$

The asymptotics of the solution in a neighborhood of the singular point ($x = 0, t = 0$) is constructed in the form of the series

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n(\zeta, \tau),$$

whose coefficients are found from the system of equations

$$\frac{\partial^2 w_0}{\partial \zeta^2} + s'(0) \frac{\partial w_0}{\partial \zeta} - \frac{\partial \varphi(w_0)}{\partial \zeta} - \frac{\partial w_0}{\partial \tau} = 0,$$

$$\begin{aligned} & \frac{\partial^2 w_n}{\partial \zeta^2} + \frac{\partial[(s'(0) - \varphi'(w_0))w_n]}{\partial \zeta} - \frac{\partial w_n}{\partial \tau} = \\ & = \frac{\partial}{\partial \zeta} G_n(w_0, \dots, w_{n-1}) - \sum_{j=1}^n \frac{\tau^j}{j!} \frac{d^{j+1}s(0)}{dt^{j+1}} \frac{\partial w_{n-j}}{\partial \zeta}, \quad n \geq 1, \end{aligned}$$

with the following the initial conditions:

$$w_n(\zeta, 0) = \begin{cases} \frac{1}{n!} \frac{d^n u_0(+0)}{dx^n} \zeta^n, & \zeta > 0, \\ \frac{1}{n!} \frac{d^n u_0(-0)}{dx^n} \zeta^n, & \zeta < 0. \end{cases}$$

It is proved that there exist infinitely differentiable for $|\zeta| + \tau > 0$ solutions of this system of initial value problems.

3 Collision of two shock waves.

Also in [3], the problem is considered in the case when the limit solution on a finite interval of time has two smooth curves of discontinuity $x = s_1(t)$ and $x = s_2(t)$, merging at the moment $t = t^*$ into one curve $x = s_3(t)$. For the solution the junction point is also singular; and in its neighborhood it is necessary to introduce the inner variables

$$\zeta = \frac{x - x^*}{\varepsilon}, \quad \tau = \frac{t - t^*}{\varepsilon},$$

where $x^* = s_3(t^*)$. Then equation (1) becomes

$$\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial \varphi(w)}{\partial \zeta} = 0.$$

The asymptotics of the solution in a neighborhood of the singular point (x^*, t^*) is constructed in the form of the series

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n(\zeta, \tau),$$

whose coefficients are found from the system of equations

$$\begin{aligned} & \frac{\partial^2 w_0}{\partial \zeta^2} - \frac{\partial \varphi(w_0)}{\partial \zeta} - \frac{\partial w_0}{\partial \tau} = 0, \\ & \frac{\partial^2 w_n}{\partial \zeta^2} - \frac{\partial[\varphi'(w_0)w_n]}{\partial \zeta} - \frac{\partial w_n}{\partial \tau} = \frac{\partial}{\partial \zeta} g_n(w_0, \dots, w_{n-1}), \quad n \geq 1. \end{aligned}$$

These equations should be supplied with conditions which are obtained as follows. The composite asymptotics, approximating the solution for $t < t^*$, is rewritten in the inner variables:

$$\begin{aligned} U_N(x, t, \varepsilon) &= \sum_{n=0}^N \varepsilon^n S_n(\zeta, \tau) + \\ &+ O\left\{\varepsilon^{n+1} \left[1 + |\tau|^{2(n+1)} (|\zeta - s'_1(t^*)\tau|^n + |\zeta - s'_2(t^*)\tau|^n)\right]\right\}, \end{aligned}$$

where functions $S_n(\zeta, \tau)$ are polynomials of degree $2n$ in τ . Thus, it is required the fulfillment of the conditions

$$\lim_{\tau \rightarrow -\infty} (w_n(\zeta, \tau) - S_n(\zeta, \tau)) = 0.$$

It is proved that solutions w_n exist and satisfy the estimates

$$\left| \frac{\partial^{l+m}}{\partial \zeta^l \partial \tau^m} (w_n(\zeta, \tau) - S_n(\zeta, \tau)) \right| < M \exp\{\mu\tau - \gamma|\zeta|\}, \quad \tau < 0,$$

$$\left| \frac{\partial^{l+m}}{\partial \zeta^l \partial \tau^m} (w_n(\zeta, \tau) - R_{3,n}(\zeta - s'_3(t^*)\tau, \tau)) \right| < M \exp\{-\gamma(\tau + |\zeta|)\}, \quad \tau > 0,$$

where M, μ, γ are positive constants depending on k, l and m ; $R_{3,n}(\zeta, \tau)$ are polynomials in τ obtained from reexpansion of the asymptotics

$$\sum_{k=0}^{\infty} \varepsilon^k v_{3,k}((x - s_3(t))/\varepsilon, t) = \sum_{k=0}^{\infty} \varepsilon^k R_{3,k}(\zeta - s'_3(t^*)\tau, \tau)$$

in a neighborhood of the shock wave $x = s_3(t)$.

4 Gradient catastrophe.

A.M. Il'in studied the case [4] when in the strip

$$\{(x, t) : t_0 \leq t \leq T, x \in \mathbb{R}\}$$

the limit ($\varepsilon = 0$) solution of the problem is a smooth function everywhere except for one smooth discontinuity curve

$$\{(x, t) : x = s(t), t \geq t^* > t_0\}.$$

A detailed presentation of his results can be found in monograph [2], where the asymptotics of the solution as $\varepsilon \rightarrow 0$ is constructed and justified with an arbitrary accuracy. Under a suitable choice of independent variables, the singular point $(s(t^*), t^*)$ coincides with the origin and in its neighborhood the following stretched variables are introduced:

$$\xi = \varepsilon^{-3/4}x, \quad \tau = \varepsilon^{-1/2}t.$$

An expansion of the solution is sought in the form of the series

$$w = \sum_{k=1}^{\infty} \varepsilon^{k/4} \sum_{j=0}^{k-1} w_{k,j}(\xi, \tau) \ln^j \varepsilon^{1/4}.$$

Substituting it into equation (1), for coefficients $w_{k,j}$ we obtain the recurrence system

$$\frac{\partial w_{1,0}}{\partial \tau} + \varphi''(0)w_{1,0} \frac{\partial w_{1,0}}{\partial \xi} - \frac{\partial^2 w_{1,0}}{\partial \xi^2} = 0,$$

$$\frac{\partial w_{k,j}}{\partial \tau} + \varphi''(0) \frac{(\partial w_{1,0} w_{k,j})}{\partial \xi} - \frac{\partial^2 w_{k,j}}{\partial \xi^2} = E_{k,j}, \quad k \geq 2.$$

These equations should be supplied with the conditions

$$w_{k,j}(\xi, \tau) = W_{k,j}(\xi, \tau), \quad \tau \rightarrow -\infty,$$

where $W_{k,j}(\xi, \tau)$ is the sum of all coefficients at $\varepsilon^{k/4} \ln^j \varepsilon^{1/4}$ in the reexpansion of the asymptotics far from the singularity (the outer expansion) in terms of the inner variables.

Investigation of solutions of this system is a central and the most laborious task in this problem. It is proved that there exist solutions $w_{k,j}(\xi, \tau)$ for $k \geq 2$, $0 \leq j \leq k-1$, infinitely differentiable for all ξ and τ .

Observe separately the properties of the leading term, which is found with the help of the Cole–Hopf transform

$$w_{1,0}(\xi, \tau) = -\frac{2}{\varphi''(0)\Lambda(\xi, \tau)} \frac{\partial \Lambda(\xi, \tau)}{\partial \xi},$$

where

$$\Lambda(\xi, \tau) = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{8}(z^4 - 2z^2\tau + 4z\xi)\right) dz$$

is a solution of the heat equation. The function $w_{1,0}$ satisfies the condition

$$w_{1,0}(\xi, \tau) = [\varphi''(0)]^{-1} H(\xi, \tau) + \sum_{l=1}^{\infty} h_{1-4l}(\xi, \tau), \quad 3[H(\xi, \tau)]^2 - \tau \rightarrow \infty,$$

where $H(\xi, \tau)$ is the Whitney fold function,

$$H^3 - \tau H + \xi = 0,$$

$h_l(\xi, \tau)$ are homogeneous functions of power l in $H(\xi, \tau)$, $\sqrt{-\tau}$ and $\sqrt{3[H(\xi, \tau)]^2 - \tau}$, which are polynomials in $H(\xi, \tau)$, τ and $(3[H(\xi, \tau)]^2 - \tau)^{-1}$.

In addition, there hold the following formulas:

$$w_{1,0}(\xi, \tau) = |\tau|^{1/2} \left(Z_0(\theta) + \sum_{j=1}^{\infty} \tau^{-2j} Z_j(\theta) \right), \quad \tau \rightarrow -\infty,$$

where $\theta = \xi|\tau|^{-3/2}$, $Z_j \in C^\infty(\mathbb{R}^1)$ are solutions of a recurrence system of ordinary differential equations;

$$w_{1,0}(\xi, \tau) = \sqrt{\tau} \left(-\frac{\text{th } z}{\varphi''(0)} + \sum_{k=1}^{\infty} \tau^{-2k} q_k(z) \right), \quad \tau \rightarrow +\infty, \quad |\xi|\tau^{1/2} < \tau^\alpha,$$

where $z = \xi\sqrt{\tau}/2$, $\alpha > 0$, $q_k \in C^\infty(\mathbb{R}^1)$ are solutions of a recurrence system of ordinary differential equations.

5 Transition of a weak discontinuity into a shock wave.

In the paper by V.G.Sushko [6], problem (1)–(2) is studied in the case when the initial function $u(x, 0, \varepsilon)$ is smooth everywhere except for one point at which it is continuous and the first derivative has a jump discontinuity. Then in some strip $t_0 \leq t \leq t^*$ the limit solution $u(x, t, 0)$ is continuous; however, the derivative u_x has a jump discontinuity, i.e., a weak discontinuity. In papers [8, 9], the behavior of the solution $u(x, t, \varepsilon)$ with the initial function

$$-(x + ax^2) \Theta(-x) (1 + q_0(x))$$

is studied near the transition point, $\Theta(x)$ is the Heaviside function.

We assume that $\varphi \in C^\infty(\mathbb{R})$, $\varphi''(u) > 0$, $\varphi(0) = \varphi'(0) = 0$, $\varphi''(0) = 1$, $\varepsilon > 0$, $a > 0$, $q_0 \in C^\infty(\mathbb{R})$, $q_0(x) = 0$ in some neighborhood of zero.

In a neighborhood of the origin ($x = 0$, $t = 0$), let us introduce the stretched variables

$$\xi = \varepsilon^{-2/3}x, \quad \tau = \varepsilon^{-1/3}t.$$

The asymptotics of the solution of the problem in a neighborhood of the origin is constructed in the form of the series

$$W = \sum_{p=2}^{\infty} \varepsilon^{p/6} \sum_{s=0}^{[p/2]-1} \ln^s \varepsilon w_{p,s}(\xi, \tau).$$

Coefficients $w_{p,s}(\xi, \tau)$ are solutions of the recurrence system

$$\begin{aligned} \frac{\partial w_{2,0}}{\partial \tau} + w_{2,0} \frac{\partial w_{2,0}}{\partial \xi} - \frac{\partial^2 w_{2,0}}{\partial \xi^2} &= 0, \\ \frac{\partial w_{3,0}}{\partial \tau} + \frac{\partial (w_{2,0} w_{3,0})}{\partial \xi} - \frac{\partial^2 w_{3,0}}{\partial \xi^2} &= 0, \\ \frac{\partial w_{p,s}}{\partial \tau} + \frac{\partial (w_{2,0} w_{p,s})}{\partial \xi} - \frac{\partial^2 w_{p,s}}{\partial \xi^2} &= \frac{\partial E_{p,s}}{\partial \xi}, \end{aligned}$$

where

$$E_{p,s} = -\frac{1}{2} \sum_{m=3}^{p-1} \sum_{l=0}^s w_{m,l} w_{p+2-m,s-l} - \sum_{q=3}^{[p/2]-s+1} \frac{\varphi^{(q)}(0)}{q!} \sum_{\substack{p_1+\dots+p_q=p+2 \\ s_1+\dots+s_q=s}} \prod_{j=1}^q w_{p_j,s_j}$$

(for $s = [p/2] - 1$ the sum in q is equal to zero), with conditions as $\tau \rightarrow -\infty$

$$w_{p,s} = \sum_{l=s}^{[p/2]-1} \frac{l!}{s!(l-s)!3^s} \ln^{l-s} |\tau| \sum_{k=\max\{1, 2l\}}^{\infty} |\tau|^{(p-3k)/2} R_{k,l,p-2l-2}(\theta)$$

in the domain

$$X^0 = \{(\xi, \tau) : |\xi| < |\tau|^{1-\gamma}, \tau < 0\} \quad (0 < \gamma < 1/2),$$

functions $R_{k,l,p-2l-2}(\theta)$ are found from the matching condition of the series W with the expansion in the boundary layer near the line of a weak discontinuity. Here, we use the notation for the self-similar variable

$$\theta = \frac{\xi}{2\sqrt{-\tau}}.$$

The leading term of the asymptotics has the form $\varepsilon^{1/3}w_{2,0}(\xi, \tau)$. The function $w_{2,0}$ is defined by formulas

$$w_{2,0}(\xi, \tau) = -\frac{2}{\Phi(\xi, \tau)} \frac{\partial \Phi(\xi, \tau)}{\partial \xi},$$

$$\Phi(\xi, \tau) = \int_0^\infty \exp\left(-\frac{4b}{3}s^3 + \tau s^2 - \xi s\right) ds, \quad b = a - \varphi'''(0)/2 > 0.$$

The next coefficient of the expansion is also obtained in an explicit form:

$$w_{3,0}(\xi, \tau) = \frac{\sqrt{\pi}}{[\Phi(\xi, \tau)]^2} \frac{\partial \Phi(\xi, \tau)}{\partial \xi}.$$

The arrangement of boundary layers breaks an analytic character of the asymptotics W at infinity that does not allow one to guess solutions of the “scattering problem” for all coefficients of the asymptotics, as was made in the problem with a smooth initial function.

6 Singularity generated by a large initial gradient.

Another type of a singular point of the solution arises in the case with two small parameters [10], when the initial condition has the form

$$u(x, 0, \varepsilon, \rho) = \nu(x\rho^{-1}), \quad x \in \mathbb{R}, \quad \rho > 0,$$

where function ν is infinitely differentiable and bounded, and ρ is the second small parameter. In [11], it is proved that in this case under conditions

$$\nu(\sigma) = \sum_{n=0}^{\infty} \frac{\nu_n^\pm}{\sigma^n}, \quad \sigma \rightarrow \pm\infty, \quad (\nu_0^- > \nu_0^+)$$

for the solution of problem (1)–(2) as $\varepsilon \rightarrow 0$ and $\mu = \rho/\varepsilon \rightarrow 0$ in the strip

$$\{(x, t) : x \in \mathbb{R}, \quad 0 \leq t \leq T\}$$

there holds the asymptotic formula

$$u(x, t, \varepsilon, \rho) = h_0\left(\frac{x}{\rho}, \frac{\varepsilon t}{\rho^2}\right) - R_{0,0,0}\left(\frac{x}{2\sqrt{\varepsilon t}}\right) + \Gamma\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) + O(\mu^{1/2} \ln \mu),$$

where

$$h_0(\sigma, \omega) = \frac{1}{2\sqrt{\pi\omega}} \int_{-\infty}^{\infty} \nu(s) \exp\left[-\frac{(\sigma - s)^2}{4\omega}\right] ds,$$

$$R_{0,0,0}(z) = \nu_0^- \operatorname{erfc}(z) + \nu_0^+ \operatorname{erfc}(-z),$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}} \int_z^{+\infty} \exp(-y^2) dy,$$

$$\sigma = \frac{x}{\rho}, \quad \omega = \frac{\varepsilon t}{\rho^2}, \quad z = \frac{\sigma}{2\sqrt{\omega}},$$

the function Γ is the solution of the equation in the inner variables ($\eta = x/\varepsilon$, $\theta = t/\varepsilon$)

$$\frac{\partial \Gamma}{\partial \theta} + \frac{\partial \varphi(\Gamma)}{\partial \eta} - \frac{\partial^2 \Gamma}{\partial \eta^2} = 0$$

with the initial condition

$$\Gamma(\eta, 0) = \begin{cases} \nu_0^-, & \eta < 0, \\ \nu_0^+, & \eta > 0. \end{cases}$$

In addition, using the renormalization method the following asymptotic formula is obtained:

$$u(x, t, \varepsilon, \rho) = \frac{1}{\nu_0^+ - \nu_0^-} \int_{-\infty}^{\infty} \Gamma\left(\frac{x - \rho s}{\varepsilon}, \frac{t}{\varepsilon}\right) \nu'(s) ds + O(\mu^{1/4}).$$

These results give asymptotics only in the leading approximation.

To construct a complete expansion near the singular point $x = 0$, $t = 0$, it is natural to “stretch” the variable x on the value ρ^{-1} . To keep the evolutionary character of equation (1), the derivative with respect to t must be the same order as the right-hand side, i.e., of order $\varepsilon \rho^{-2}$. Thus, we make the change of variables

$$x = \rho \sigma, \quad t = \frac{\rho^2}{\varepsilon} \omega.$$

The inner expansion is sought in the form of the series

$$H(\sigma, \omega, \mu) = \sum_{n=0}^{\infty} \mu^n h_n(\sigma, \omega), \quad \mu = \frac{\rho}{\varepsilon} \rightarrow 0.$$

Substituting it into the equation

$$\frac{\partial h}{\partial \omega} - \frac{\partial^2 h}{\partial \sigma^2} = -\mu \frac{\partial \varphi(h)}{\partial \sigma},$$

for $h(\sigma, \omega) \equiv u(\rho \sigma, \rho^2 \omega / \varepsilon)$, we obtain a recurrence chain of the initial value problems

$$\begin{aligned} \frac{\partial h_0}{\partial \omega} - \frac{\partial^2 h_0}{\partial \sigma^2} &= 0, & h_0(\sigma, 0) &= \nu(\sigma), \\ \frac{\partial h_1}{\partial \omega} - \frac{\partial^2 h_1}{\partial \sigma^2} &= -\frac{\partial \varphi(h_0)}{\partial \sigma}, & h_1(\sigma, 0) &= 0, \\ \frac{\partial h_n}{\partial \omega} - \frac{\partial^2 h_n}{\partial \sigma^2} &= -\frac{\partial E_n}{\partial \sigma}, & h_n(\sigma, 0) &= 0, \end{aligned}$$

where

$$E_n = \sum_{q=1}^{n-1} \frac{\varphi^{(q)}(h_0)}{q!} \sum_{n_1 + \dots + n_q = n-1} \prod_{p=1}^q h_{n_p}, \quad n \geq 2.$$

It follows that all coefficients $h_n(\sigma, \omega)$ are uniquely determined:

$$h_0(\sigma, \omega) = \frac{1}{2\sqrt{\pi\omega}} \int_{-\infty}^{\infty} \nu(s) \exp \left[-\frac{(\sigma - s)^2}{4\omega} \right] ds,$$

$$h_n(\sigma, \omega) = - \int_0^{\omega} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(\omega - v)}} \exp \left[-\frac{(\sigma - s)^2}{4(\omega - v)} \right] \frac{\partial E_n}{\partial s} ds dv.$$

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